**A minimum problem**

Find the shortest distance from the origin O to the curve $y=\frac{1}{x^{4}}$ (where x > 0).

Use the following methods:

**(1) Calculus,**

**(2) A.M.≥ G.M.**





 **(1) Calculus:**

 **Method 1**

 $s=AB=\sqrt{\left(x-0\right)^{2}+\left(y-0\right)^{2}}=\sqrt{x^{2}+y^{2}}=\sqrt{x^{2}+\left(\frac{1}{x^{4}}\right)^{2}}=\sqrt{x^{2}+\frac{1}{x^{8}}}$

 $s^{2}=x^{2}+\frac{1}{x^{8}}$

 $2s\frac{ds}{dx}=2x-\frac{8}{x^{9}}⇒\frac{ds}{dx}=\frac{1}{s}\left(\frac{x^{10}-4}{x^{9}}\right)$

 For critical points, $\frac{ds}{dx}=0$.

 $x^{10}-4=0⟹ x=\sqrt[10]{4}=\sqrt[5]{2}≈1.148698354997$

 For $0<x<\sqrt[5]{2}, \frac{ds}{dx}<0$. and for $x>\sqrt[5]{2}, \frac{ds}{dx}<0$.

 Therefore y is a minimum when $x=\sqrt[5]{2}$.

 When $x=\sqrt[5]{2}, $

 $s=\sqrt{\left(\sqrt[5]{2}\right)^{2}+\frac{1}{\left(\sqrt[5]{2}\right)^{8}}}=\sqrt{\frac{\left(\sqrt[5]{2}\right)^{10}+1}{\left(\sqrt[5]{2}\right)^{8}}}=\sqrt{\frac{4+1}{\sqrt[5]{256}}}=\sqrt{\frac{5}{\sqrt[5]{256}}}=\sqrt[10]{\frac{3125}{256}}≈1.2842838037078$

 **Method 2**

**** Construct a circle centre origin and radius r : $x^{2}+y^{2}=r^{2}$ …. (1)

Common tangent

Too big circle, two intersection points

Too small circle, no intersection point

 In order to find the shortest r, we like to have this circle **touches** the given curve:

 $y=\frac{1}{x^{4}}$ …. (2)

 Then (1) and (2) should have a common tangent

 at the point of contact .

 Differentiate (1) and (2),

$$\left\{\begin{array}{c} 2x+2y\frac{dy}{dx}=0 ⟹ \frac{dy}{dx}=-\frac{x}{y}\\\frac{dy}{dx}=-\frac{4}{x^{5}}\end{array}\right.$$

 $∴ -\frac{x}{y}=-\frac{4}{x^{5}} ⟹ y=\frac{x^{6}}{4}$ …. (3)

 By (2), $ \frac{1}{x^{4}}=\frac{x^{6}}{4} ⟹ x^{10}=4 ⟹ x=\sqrt[10]{4}=\sqrt[5]{2}≈1.148698354997$

 By (3), $y=\frac{\left(\sqrt[5]{2}\right)^{6}}{4}$

 By (1), $r^{2}=x^{2}+y^{2}=\left(\sqrt[5]{2}\right)^{2}+\left[\frac{\left(\sqrt[5]{2}\right)^{6}}{4}\right]^{2}≈1.6493848884661$

 $r≈1.2842838037078$

 **(2) A.M.≥ G.M. :**

 $s=AB=\sqrt{\left(x-0\right)^{2}+\left(y-0\right)^{2}}=\sqrt{x^{2}+y^{2}}=\sqrt{x^{2}+\left(\frac{1}{x^{4}}\right)^{2}}=\sqrt{x^{2}+\frac{1}{x^{8}}}$

 $=\sqrt{\frac{x^{2}}{4}+\frac{x^{2}}{4}+\frac{x^{2}}{4}+\frac{x^{2}}{4}+\frac{1}{x^{8}}}\leq \sqrt{5\sqrt[5]{\left(\frac{x^{2}}{4}\right)\left(\frac{x^{2}}{4}\right)\left(\frac{x^{2}}{4}\right)\left(\frac{x^{2}}{4}\right)\left(\frac{1}{x^{8}}\right)}}$ (A.M. ≥ G.M.)

 $=\sqrt{5 \sqrt[5]{\frac{1}{256}}}=\sqrt[10]{\frac{3125}{256}}≈1.2842838037078$

The point is to get rid of all x !

 where equality holds if and only if

 $\frac{x^{2}}{4}=\frac{1}{x^{8}} ⇔ x^{10}=4 ⇔ x=\sqrt[10]{4}=\sqrt[5]{2}≈1.148698354997$

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